Low-to-high frequency targeted energy transfer using a nonlinear energy sink with softening-hardening nonlinearity

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A B S T R A C T

This paper investigates the performance of a nonlinear energy sink (NES) equipped with a softening–hardening (SH) element for mitigating vibrations that occur below its linear, low-energy frequency. First, an SH elastic structure is constructed, tested quasi-statically, and the resulting restoring force is modeled using a polynomial. Next, the SH polynomial is non-dimensionalized and a theoretical system composed of a linear oscillator (LO) and NES is studied where the NES has a linear, low-energy frequency above that of the LO. The underlying nonlinear normal modes (NNMs) of the system are studied by using harmonic excitation and tracking the changes in the frequency response functions. The transient performance of the NES is investigated and compared with that of a NES with the same linear stiffness but only a cubic nonlinearity. The theoretical performance of the NES is verified using a comparable experimental system that incorporates the SH elastic structure built in the beginning of the paper. The results of this work demonstrate that an SH NES is capable of mitigating vibrations that occur at frequencies below its own linear, low-energy frequency.

1. Introduction

Nonlinearity is present in nearly all structures, and this nonlinearity causes the dynamics of the system to change based on the energy stored in the system. Specifically, the frequencies and mode shapes governing the motion of the structure change as the energy in the system varies. In this sense, the system can be said to possess nonlinear normal modes (NNMs) [1,2], which are time-periodic oscillations that are not necessarily synchronous and represents a nonlinear extension of linear normal modes. This distinction makes nonlinear systems more complicated than their linear counterparts [3] and introduces phenomena, such as internal resonances, that do not arise under linear vibrations.

One of the most prolific implementations of nonlinearity is the phenomena of targeted energy transfer (TET), which involves the irreversible transfer of energy from a primary linear structure to a series of local, nonlinear attachments called nonlinear energy sinks (NESs) [4–6]. However, in the typical application of TET, the NES is equipped with a strong hardening nonlinearity and, thus, must have a linear (low energy) frequency that is less than that of the mode of the primary linear structure that it is intended to absorb energy from [3,7,8]. In other words, if the linear frequency of the NES is above that of a mode in the primary structure, then the NES will not be able to dissipate energy in that mode and, instead, acts as a mass effect and lowers that mode's natural frequency. This behavior can be seen in [9] where an NES installed on a model airplane wing was tuned to interact with the second mode of the wing and, thus, was unable to mitigate the response of the first mode. Moreover, the design of nonlinearities in real NESs often result in a weak linear stiffness, such that the NES has a non-zero frequency and lowering that linear frequency often comes with sacrificing the strength of the nonlinearity. In the case of highly flexible structures that possess multiple modes at low frequencies (below 10 Hz), an NES with even a weak linear stiffness may be unable to mitigate the motion of the lowest flexible modes.

To overcome this limitation, many efforts have been accomplished, one of which is applying quasi-zero stiffness (QZS) to achieve passive energy isolation in low dynamic regime [10–13]. In this context, QZS can be obtained by combining a positive stiffness component with a negative stiffness adjustment, resulting in a combination of linear, plate, and hardening segments as seen in Fig. 1. In practice, QZS isolators can be achieved in many mechanisms: cam-roller [14–16], buckled beams [17–20] and many other structures [21–25]. By applying QZS, a softening–hardening (SH) process is introduced to the NNMs of the system, which will enable the NES to interact with and mitigate the modes of the primary structure that have frequencies below the linear, low-energy frequency of the attachment.

In this work, we posit that an NES with an SH spring will be able to interact with and mitigate vibrations that occur at a lower frequency than the linear frequency of the NES. The reason is that the transition into and the plateau region of the SH spring produces a softening effect that reduces the frequency of the NES. The reduction in frequency allows the NES to interact with modes that have linear frequencies

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below that of the NES’s linear frequency. Next, we aim to evaluate this hypothesis computationally and experimentally as well as the performance of an SH NES with a linear frequency below the targeted mode for mitigation. The paper is organized as follows. Section 2 presents a design of the SH spring based on an elastic strut element proposed by Bunyan and Tawfick [26] followed by the experimental characterization and modeling of its restoring force. Section 3 presents a computational study into the effectiveness of an NES equipped with an SH spring at mitigating the motion of a linear oscillator (LO), and comparisons are made with a comparable NES with only a cubic stiffness nonlinearity. Section 4 presents the experimental validation of the computational results for the NES with SH spring. Finally, Section 5 presents concluding remarks about this work.

2. Design, identification, and nondimensionalization of softening-hardening element

2.1. Elastic strut elements

In this work, we consider the elastic strut elements created by Bunyan and Tawfick [26] as shown in Fig. 2, which consists of a triangular shaped strut that undergoes buckling when compressed. The elastic strut elements were designed using Autodesk Inventor Professional and include two bolt holes for clamping onto structures. The CAD model is shown in Fig. 2(a). The strut has a width of 0.0221 m, a height of 0.0193 m, and a thickness of 0.00483 m. The arms have widths of 0.0178 m and are chosen based on the design described by Bunyan and Tawfick [26]. A dimensioned drawing of the elastic strut is provided as supplementary material.

To manufacture the springs, master patterns of the spring were created with an SLA 3D printer (Form 3, Formlabs Inc) to prevent adhesion of silicone to the masters during the next step. A duplication silicone (Elite Double 32, Zhermack) was then cast around the master patterns to create negative molds of the spring geometry. The master patterns were removed from the silicone negatives after the silicone was set. To create the springs, a urethane rubber was prepared by combining part A to B at a 1:1 mass ratio and then mixing in a planetary mixer for 2 min (SpeedMixer DAC 400.2 VAC, FlackTek Inc). Mold release spray was applied to the silicone negatives to prevent the rubber from adhering to the molds. The uncured rubber was then cast into the silicone negatives and cured under 40 psig of pressure at room temperature for 24 h. Fig. 2(b) demonstrates the completed spring element.

![Fig. 1. A typical compression plot of a softening-hardening (SH) process, and it is composed of 3 parts: a linear, plateau, and stiffening part.](image)

\[
F(x) = (a_1 |x| + a_2 |x|^2 + a_3 |x|^3 + a_4 |x|^4 + a_5 |x|^5) \text{ sgn}(x).
\]  

(1)

This model is chosen because it provides a symmetric restoring force under both compression and tension, and a symmetric restoring force will be used in the theoretical study and in the experimental system. The parameters were identified using the Curve Fitting toolbox in MATLAB® and the dimensional values are provided in Table 1. The model restoring force is plotted on top of the measured force and the resulting R-squared value is 0.9907 between the model and the experiment (Fig. 4(b)). We determined that the fifth-order model was optimal by fitting third-order through ninth-order polynomials using the same procedure as the fifth-order, which revealed that the fifth-order was the lowest order where a considerable increase in R-squared value occurred. The R-squared values for all of the polynomial models computed using the 1 mm/min measurement.

<table>
<thead>
<tr>
<th>Order</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>R²</td>
<td>0.9756</td>
<td>0.9838</td>
<td>0.9907</td>
<td>0.9909</td>
<td>0.991</td>
<td>0.9912</td>
<td>0.9912</td>
</tr>
</tbody>
</table>

2.2. Measurement and identification of strut element restoring force

The compressive behavior of each element was then characterized through a series of compression tests using a universal testing machine (Instron 5966) with a 200 N load cell. To aid in the testing, a plastic mount was 3D printed and installed in the Instron machine’s grip to provide a smooth surface for compressing the strut. The mount is shown in Fig. 3(a). The compression tests consisted of 5 pre-cycle compression tests to eliminate any initial hysteresis of the material and 3 formal cycles with an 8 mm maximum displacement. The compression tests were completed for testing speeds of 10 mm/min, 5 mm/min, and 1 mm/min and the resulting restoring forces are depicted in Fig. 4.

As shown in Fig. 4, the restoring force is observed to be similar for all three testing speeds and there is minimal hysteresis between the loading and unloading curve. As such, the spring can be approximated as elastic and only one of the three speeds is needed to identify a model for the restoring force. To this end, we use the third cycle from the 1 mm/min loading speed to fit a fifth-order polynomial to the restoring force in the following form:

\[ R^2 = 0.9912 \]

3. Theoretical study

3.1. System model and nondimensionalization

We consider the dynamics of a linear oscillator (LO) with a nonlinear energy sink (NES) installed on it. The LO is linearly coupled to the ground whereas the NES is nonlinearly coupled to the LO using the SH element (i.e., the polynomial model in Eq. (1)). The system is depicted in Fig. 5 and the governing equations of motion are

\[ m_1 \ddot{x} + d_1 \dot{x} - d_4 \ddot{\delta} + k_1 x = \sum_{i=1}^{n} a_i |\delta|^n \text{ sgn}(\delta) = 0, \]

(2)
Fig. 2. (a) CAD model of the elastic strut elements and (b) the corresponding experimental part.

Fig. 3. (a) Configuration of the compression test on Instron machine with the 3D-printed mount and photographs of the (b) linear phase, (c) plateau (softening) phase, and (d) stiffening phase of the elastic strut element.

Fig. 4. (a) Compression curve of the elastic strut elements under testing speeds of 10, 5 and 1 mm/min. (b) Curve fitting of the compression curve of the elastic strut elements. Red line: experimental compression curve (1 mm/min); blue line: fitting curve of the compression curve using 5th order polynomial. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)
original dimensional parameters through the following relations

\[
\epsilon = \frac{m_1}{m_2}, \quad \lambda_k = \frac{d_k}{\sqrt{m_1 k_x}}, \quad \lambda_c = \frac{d_c}{\sqrt{m_1 k_x}},
\]

and the SH spring parameters are given by

\[
\beta_1 = \frac{a_1}{k_x}, \quad \beta_2 = \frac{a_2}{\sqrt{k_x a_5}}, \quad \beta_3 = \frac{a_3}{a_4}, \quad \beta_4 = a_4 \sqrt{\frac{k_x}{a_5}}, \quad \beta_5 = \frac{a_4 k_x}{a_5^2}.
\]

The dimensionless parameters for the SH spring are provided in Table 1 and the restoring force is plotted in Fig. 6 along with the restoring force of System II discussed in the next subsection.

3.2. Parameter selection

As stated previously, we hypothesize that an SH NES could mitigate vibrations that occur at frequencies below its linear frequency. We consider two versions of the same system: System I incorporates the SH spring element using the parameters listed in Table 1 and System II includes only a cubic stiffness nonlinearity. Specifically, the equation of motion governing the NES in System II is

\[
\epsilon \ddot{\delta} + \dot{\delta} + \sum_{n=1}^{5} \alpha_n |\delta|^n \text{sgn}(\delta) = 0,
\]

where \(\delta(t) = y(t) - x(t)\). The equations of motion are nondimensionalized into the following forms

\[
\ddot{\xi}_1 + \dot{\xi}_1 + \lambda_1 \xi_1 - \lambda_2 \dot{\xi}_2 + \sum_{n=1}^{5} \beta_n |\rho|^n \text{sgn}(\rho) = 0,
\]

\[
\epsilon \ddot{\xi}_2 + \dot{\xi}_2 + \beta_3 \dot{\rho} + \sum_{n=1}^{5} \beta_n |\rho|^n \text{sgn}(\rho) = 0,
\]

where \(\rho(t) = \dot{\xi}_2(t) - \dot{\xi}_1(t)\). The resulting parameters are related to the original dimensional parameters through the following relations

\[
\epsilon = \frac{m_1}{m_2}, \quad \lambda_k = \frac{d_k}{\sqrt{m_1 k_x}}, \quad \lambda_c = \frac{d_c}{\sqrt{m_1 k_x}},
\]

and

\[
\beta_1 = \frac{a_1}{k_x}, \quad \beta_2 = \frac{a_2}{\sqrt{k_x a_5}}, \quad \beta_3 = \frac{a_3}{a_4}, \quad \beta_4 = a_4 \sqrt{\frac{k_x}{a_5}}, \quad \beta_5 = \frac{a_4 k_x}{a_5^2}.
\]

The dimensionless parameters of the two systems considered.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>System I</th>
<th>System II</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\epsilon)</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>(\lambda_k)</td>
<td>0.02</td>
<td>0.02</td>
</tr>
<tr>
<td>(\lambda_c)</td>
<td>0.02</td>
<td>0.02</td>
</tr>
<tr>
<td>(\beta_1)</td>
<td>0.6271</td>
<td>0.6271</td>
</tr>
<tr>
<td>(\beta_2)</td>
<td>-1.241</td>
<td>0</td>
</tr>
<tr>
<td>(\beta_3)</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(\beta_4)</td>
<td>-0.3408</td>
<td>0</td>
</tr>
<tr>
<td>(\beta_5)</td>
<td>0.0438</td>
<td>0</td>
</tr>
</tbody>
</table>

Note that in both systems, the NES has a linear frequency higher than that associated with the LO. The nondimensional parameters of each system are provided in Table 3. Among these parameters, those governing the LO are selected based on typical nondimensional values in previous studies [27,28]. Note that, the coupling damper, \(\lambda_c\), is assumed to be equal to the grounding damper, \(\lambda_g\), because the SH element is made from polyurethane and is expected to introduce large damping compared to a metallic component (e.g., wires undergoing large deformations).

3.3. Nonlinear normal mode study

To build an understanding for the dynamics of this system, we consider the NNMs that govern the frequency transitions that arise due to the nonlinearity. Based on the definition provided earlier, one approach to investigating the NNMs is to plot how the frequency-response functions (FRFs) change as the amplitude or energy of the system is varied [1,29–34]. To this end, we consider the response of the system under harmonic excitation, in the form \(F(t) = P \sin(\omega t)\), applied directly to the LO with zero initial conditions for a total non-dimensional time of 5000 with a step size of 2e–2. We considered 500 different forcing amplitudes logarithmically spaced from 10^{-4} to 10^{2} and 501 different frequencies linearly spaced from 0.1 to 10.

For each simulation, we compute the FRFs by dividing the maximum displacement of the LO and NES in the time window \([4800, 5000]\) by the input force \(P\). This window is chosen to ensure that the transient response has fully died out and that the maximum displacement is due to the harmonic forcing only. We choose the maximum displacement because the response may not exhibit a steady-state amplitude or even a periodic response when internal resonances are activated in the dynamics. Nevertheless, this approach provides a good window into how the frequency behavior of the system evolves as the amplitude (energy) increases.

We depict the resulting FRFs for the LO, NES, and the relative displacement in Fig. 7 with the receptance shown as a colormap with darker colors representing low receptance and lighter colors representing high receptance. At low forcing amplitudes (below 0.1), the response of the system is governed by two nearly linear modes of vibrations as shown by the two horizontal bands of high receptance. In this case, the low-frequency band represents the first NNM and corresponds to in-phase motion of both masses at a frequency of 0.94 with the NES moving nearly the same amount as the LO (the corresponding linear, mass-orthonormalized mode shape is given by [0.938, 1.095] with the first and second entries corresponding to the LO and NES amplitude, respectively). The high-frequency band represents the second NNM and corresponds to out-of-phase motion of the two masses at a frequency of 2.64, such that the second NNM has a low-energy, linear frequency 2.81 times that of the first NNM. Note that, unlike typical NESs with negligible linear stiffness, the installation of the NES on the LO introduces the second NNM into the dynamics of the system and this is confirmed by the fact that the response is dominated by motion in the NES at frequencies around the second NNM. Moreover, the corresponding mass-orthonormalized mode shape...
with each other through internal resonances, such that we can observe

corresponds to a SMR indicating that the LO and NES are interacting

in-phase motion and point ii showing out-of-phase motion. Point iii

response of the first and second NNMs, respectively, with point i showing

interactions arising for weak forcing ($P = 0.06$) at the linear frequency

of the first NNM ($\omega = 0.95$). Point iv depicts out-of-phase motion similar
to the motion of point ii, which implies that the internal resonances are
facilitated by the first NNM at this forcing level. Looking at point v, which is at the same frequency as point iv but at a magnitude of
forcing higher, we find that the motion has returned to a simple in-

phase motion as observed in point i. In fact, the response in point v is nearly identical to that in point iv except for the difference in
amplitude and a slight difference in phase. Points vi and vii exhibit SMR
similar to point iii indicating that the LO is transferring energy into the
NES through internal resonances. Point viii exhibits nearly out-of-phase
motion, which indicates that this point lies on a non-interacting branch
of the second NNM whereas points vi and vii lie on interacting parts of
the branch. Point ix depicts in-phase motion with the NES moving more
than the LO, such that the portion of the FRF that fans downwards for
$P > 1$ corresponds to in-phase motion with no nonlinear interactions
between the LO and NES. Point x shows in-phase motion with the
LO and NES having nearly the same amplitude, which indicates that
the bright horizontal band now exhibits nearly linear motion. Finally,
points xi and xii depict quasi-periodic or aperiodic motions indicating
that the system responds chaotically within the band of increasing
frequency of the second NNM.

3.4. Transient performance of SH NES

To investigate the performance of the SH NES, we consider the
response of the system under the initial conditions $\dot{\xi}_1(0) = \xi_1(0) =
0$ and $\dot{\xi}_2(0) = \xi_2(0) = \dot{v}_0$, such that an initial velocity is applied to

only the LO, which is equivalent to exciting the LO using an impulsive
force. We simulated the response of the system for 1000 logarithmically
spaced initial velocities for the range $v_0 \in [10^{-3}, 10^3]$, which provides
a large enough range to probe the entire dynamics of the system. The
response was simulated using $ode45$ in MATLAB\textsuperscript{®} for a time step of
0.01 for a total time of 3000. Additionally, the relative and absolute
tolerances of the solver were both set to 10\textsuperscript{-12}. This combination of
parameters ensures that the system can be integrated accurately at
high initial velocities. We consider two different metrics for assessing
the performance of the SH NES: first, we compare the percent energy

for the corresponding linear second mode is $[-0.346, 2.967]$, such
that the motion of the NES more than 8.5 times that of the LO when
the second mode is excited. Above a force of 0.01, the second NNM
decreases in frequency until it appears to intersect the first NNM. The
decrease in frequency is due to the buckling of the spring that results
in a softening-type nonlinearity. The intersection occurs at a forcing
amplitude of approximately 0.3 and after this force, the frequency of
the second NNM increases away from that of the first NNM.

The decrease and subsequent increase in frequency of the second
NNM leads to the possibility of internal resonances where energy is
pumped from the first NNM into the second NNM. The result of such
internal resonances is the presence of strongly modulated responses
(SMRs) [3] where the LO and NES exhibit strong beating behavior when
forced harmonically. When excited impulsively, the internal resonances
would cause the motion of the LO to rapidly decay resulting in a
dramatic reduction in its settling times. To investigate potential internal
resonances, we selected twelve combinations of forcing amplitude and
frequency as shown on the FRFs in Fig. 7, and present the correspond-
ing displacements of the LO and NES in Fig. 8. Note that the time axis
on each plot in Fig. 8 has been chosen to capture the behavior clearly on
an individual basis. The forcing frequency and amplitude combinations
are reported in Table 4.

<table>
<thead>
<tr>
<th>Point</th>
<th>Frequency, $\omega$</th>
<th>Force, $P$</th>
<th>Point</th>
<th>Frequency, $\omega$</th>
<th>Force, $P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
<td>0.95</td>
<td>0.005</td>
<td>vii</td>
<td>1.25</td>
<td>0.372</td>
</tr>
<tr>
<td>ii</td>
<td>2.63</td>
<td>0.005</td>
<td>viii</td>
<td>1.4</td>
<td>0.372</td>
</tr>
<tr>
<td>iii</td>
<td>0.95</td>
<td>0.06</td>
<td>x</td>
<td>0.7</td>
<td>2</td>
</tr>
<tr>
<td>iv</td>
<td>2.377</td>
<td>0.06</td>
<td>x</td>
<td>0.95</td>
<td>2</td>
</tr>
<tr>
<td>v</td>
<td>0.95</td>
<td>0.372</td>
<td>xi</td>
<td>1.35</td>
<td>2</td>
</tr>
<tr>
<td>vi</td>
<td>1.1</td>
<td>0.372</td>
<td>xii</td>
<td>1.75</td>
<td>2</td>
</tr>
</tbody>
</table>

Fig. 7. FRF of steady state response of System 1 under harmonic excitation with varying forcing amplitude and frequency. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)
dissipated by each damper in the system, which is a common approach to determining the performance of an NES [35–37] second, we compute the 50%, 10%, 5%, and 2% settling times of the LO, which allows us to see how the amplitude of the LO varies across all initial velocities considered.

The energy dissipated by each damper are computed as

\[ D_{LO} = \int_0^T \lambda \dot{\xi}^2 \, dt, \]  
\[ D_{NES} = \int_0^T \lambda \left( \ddot{\xi}_2 - \ddot{\xi}_1 \right)^2 \, dt. \]  

These are normalized by dividing by the initial mechanical energy, \( \frac{1}{2} m_1 v_0^2 \), then multiplied by 100% to find the percent energy dissipated by each damper. The resulting percent energies dissipated for System I and II are presented in Fig. 9 for all initial velocities considered.

At initial velocities at and below 10\(^{-2}\), the behaviors of both systems are identical with the grounding damper dissipating 86% of the total energy and the coupling damper only dissipates 14%. The reason is that, at such low initial velocities, both systems exhibit approximately linear behavior and the response of the system is dominated by motion in the LO. Starting with initial velocities of 10\(^{-2}\), the behavior of System I diverges from that of System II with the amount of energy dissipated by the coupling damper increasing. The amount of energy dissipated by the coupling damper increasing gradually initially, then rapidly increases after an initial velocity of \( \sim 0.59 \) with the two dampers dissipating the same amount of energy at an initial velocity of \( \sim 0.761 \).

The coupling damper dissipates the majority of the energy in the system for the range of initial velocities of [0.761, 6.163], indicating that most of the energy is irreversibly transferred into the NES from the LO inside this range. The two dampers again dissipate equal amounts of energy at an initial velocity of \( \sim 6.163 \) and, for higher initial velocities, the grounding damper dominates the dissipation of energy. Furthermore, at high initial velocities, the behavior of System I begins to converge to that of System II, which indicates a second regime of linear response at high energies.

Fig. 9 reveals that the LO-NES system coupled by SH spring has two main advantages in energy dissipation over the corresponding cubic nonlinearity LO-NES system: (1) the SH element allows an NES with linear frequency higher than the primary mode to interact with that mode and parasitically absorb energy from it whereas a typical cubic NES is unable to do so; and (2) the use of an SH element results in an increased range where the NES is able to mitigate the motion of the primary structure. The second advantage is due to the softening–hardening behavior (for increasing energy) that arises due to the SH element in the proposed system. Furthermore, we note that the coupling damper in System II never dominates the dissipation of energy in the system, which indicates that the NES is unable to parasitically absorb energy from the LO for any of the initial velocities considered.

To further investigate the performance of the SH and cubic NESs, we also computed the 50%, 10%, 5%, and 2% settling times of the LO for the same simulations performed to compute the percent energy dissipated by each damper in the system.
dissipated by each damper as discussed previously [38,39]. The settling times are computed as the final time where the LO achieves 50%, 10%, 5%, and 2% of its maximum displacement amplitude. We chose these four settling percentages to probe how the NES affects the response of the LO in across the entire response window. The resulting plots are presented in Fig. 10 and provide a comparison between the settling times for System I and System II. Similar to Fig. 9, we find that the settling times for both systems agree at low velocities, diverge for a middle range of velocities (roughly 0.1 to 100), then converge to approximately the same time at high initial velocities. The divergence between the two systems arises due to strong decrease in the settling time of the LO in System I, which corresponds to the regime of increased dissipation by the coupling damper. Thus, the reduction of settling times correspond to a irreversible transfer of energy from the first NNM to the second NNM, where the transferred energy becomes trapped and is rapidly dissipated by the NES. Unlike System I, we find that System II has only a minimal effect on the settling time of the LO, which further confirms that the NES in System II is unable to mitigate the motion of the LO.

4. Experimental study and results

4.1. System design and construction

To experimentally assess the performance of an NES with an SH spring, we constructed a physical representation of System I studied in Section 3 and the resulting system is depicted in Fig. 11. The LO consists of a base platform, two mounting blocks for the SH springs, and one mounting block for coupling the NES to the LO using thin steel flexures. The LO base platform is constructed from an aluminum plate of dimensions 0.1524 m × 0.1524 m × 0.0127 m. The LO is grounded to an optical table using two aluminum L-shaped brackets, thin steel flexures, and 10–32 UNF bolts. The two thin, narrow steel flexures using 6–32 bolts. The portion of these flexures that undergoes bending when the NES moves have dimensions of 0.0584 m × 0.0102 m × 0.000127 m. The NES is nonlinearly coupled to the LO through the SH strut that is mounted onto the LO directly. The mass of completely assembled LO is 0.905 kg and the mass of the NES is 0.095 kg, such that the NES has only 10.5% the mass of the LO.

4.2. Experimental measurements and system identification

In all experiments, we applied an impulsive excitation to the LO directly using a PCB Piezotronics modal hammer (model 086C03) with a black rubber tip installed. We measured the accelerations of the LO and NES using PCB accelerometers (model 352C03) with an average sensitivity of 10 mV/(m/s²) at a sampling frequency of 4096 Hz for 32 s. The measurements were acquired using a Data Physics Abacus 906 data acquisition system and the Data Physics Software Suite (Data Physics, San Jose, CA, USA). The accelerations were then integrated numerically, and high-pass filtered using a fifth-order Butterworth filter with a cutoff frequency of 4 Hz to obtain the corresponding velocity response. This procedure was then applied to the velocities to obtain the displacement response of both the LO and NES.

To aid in the experimental verification, we first tested the LO without the NES installed and used the resulting data to identify a mathematical model for the LO. In this case, the LO acts as a simple harmonic oscillator governed by the equation of motion

\[ m_{LO} \ddot{x} + d_0 \dot{x} + k_0 x = F(t). \]  

We used the resulting response for an impact of 63 N and computed the corresponding FRF for the LO using the integrated displacements. The parameters of the LO were identified using the half-power method, which resulted in \( d_1 = 1.310 \text{ N/m/s} \) (\( \zeta = 0.00762 \)) and \( k_1 = 8173.3 \text{ N/m} \) (\( \omega_n = 95.03 \text{ rad/s}, f_s = 15.125 \text{ Hz} \)). We present a comparison between the FRF computed using the experimental data and the analytical FRF computed using the identified parameters and Eq. (11) in Fig. 12.

Following the identification of the LO, the NES was installed using the thin steel flexures and the SH struts were not installed, such that the resulting system represents a two DOF linear system. In this configuration, an impact was applied directly to the NES using a small PCB modal hammer (model 086E80) and the resulting response was used to identify the stiffness of the thin steel coupling flexures. From these measurements, it was determined that the NES oscillated at a frequency of 2.875 Hz, such that the frequency of the thin steel flexures is approximately 31 N/m (negligible compared to the linear stiffness of the SH strut, which was identified as 4918 N/m in Section 2). Incorporating the linear model for the LO and linear stiffness from the SH strut and steel flexures, the resulting natural frequencies of the underlying linear system are \( \omega_1 = 14.28 \text{ Hz} \) and \( \omega_2 = 38.52 \text{ Hz} \) corresponding to mass-orthonormalized mode shapes of \( \phi_1 = [0.982, 1.161] \) and \( \phi_2 = [-0.376, 3.030] \). The ratio of the frequency second mode to that of the first mode 2.6981, which is comparable to the ratio of 2.81:1 found for the theoretical system. The mass-orthonormalized mode shapes of the experimental system also agree well with those of the theoretical system.

After identifying the stiffness of the linear coupling flexures, the SH strut elements were installed, then 15 different impacts ranging from 6 N to 985 N were applied to the LO and the resulting responses were measured using the settings described previously. Since a model for the elastic restoring force SH strut element was identified based on the compression tests discussed in Section 2, the only parameter remaining to be identified is the coefficient of the coupling damper. To this end, we used the responses measured for an impact of 433.5 N to identify the damping coefficient, \( d_s \), using a time series optimization approach used previously in [27,40]. This procedure optimizes the unknown parameters by minimizing the root mean squared error (RMSE) between the experimentally measured and computationally simulated...
responses for both the LO and NES. We implemented this procedure using patternsearch in MATLAB®, which is a global optimization method based on direct search, with an initial guess of $d_c = 1$ N s/m with lower and upper limits of 0 and 5 N s/m, respectively. We used the following settings for the optimization routine: mesh tolerance of $10^{-12}$, function tolerance of machine epsilon ($2.2204 	imes 10^{-16}$), tolerance on variable changes of machine epsilon, maximum function evaluations of $10^{10}$, and maximum iterations of $10^{10}$. This combination of parameters ensures that the optimization routine stops once the mesh-size drops below $10^{-12}$ and makes the routine independent of the initial guess, such that it converges to the global minimum. Using this approach, the optimization routine identified that $d_c = 2.543$ N s/m with the RMSE being 0.00028 and 0.00050 for the LO and NES, respectively. The final physical parameters are provided in Table 5 along with their corresponding non-dimensional values and the non-dimensional values of the theoretical system. We present the resulting comparison between the experimentally measured and computationally simulated responses in Fig. 13(a) and (b) for the LO and NES, respectively. The relative disagreement between the measurement and simulation is due to the simplified model chosen for the SH strut element; however,
### Table 5
Parameters of the LO-NES system with dimensional and nondimensional values.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Dimensionless values</th>
<th>Dimensional values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>- Theoretical system</td>
<td>- Experimental system</td>
</tr>
<tr>
<td>$m_1$</td>
<td>1</td>
<td>0.905 kg</td>
</tr>
<tr>
<td>$m_2$</td>
<td>0.1</td>
<td>0.095 kg</td>
</tr>
<tr>
<td>$k_g$</td>
<td>1</td>
<td>8187.7 N/m</td>
</tr>
<tr>
<td>$k_c$</td>
<td>0.0038</td>
<td>31 N/m</td>
</tr>
<tr>
<td>$d_g$</td>
<td>0.0152</td>
<td>1.310 N/m</td>
</tr>
<tr>
<td>$d_c$</td>
<td>0.0296</td>
<td>2.543 N/m</td>
</tr>
<tr>
<td>$a_1$</td>
<td>0.6271</td>
<td>4918 N/m</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$-4.723 \times 10^6$ N/m^2</td>
<td></td>
</tr>
<tr>
<td>$a_3$</td>
<td>0.6494</td>
<td>1.847 $\times 10^8$ N/m^3</td>
</tr>
<tr>
<td>$a_4$</td>
<td>$-3.055 \times 10^{11}$ N/m^4</td>
<td></td>
</tr>
<tr>
<td>$a_5$</td>
<td>0.0184</td>
<td>1.903 $\times 10^{13}$ N/m^5</td>
</tr>
</tbody>
</table>

The agreement is enough for us to perform predictions regarding the performance of the NES as done in Section 3.

#### 4.3. Experimental verification of proposed NES

To investigate the performance of the experimental SH NES, we simulated the response of the model system under the initial conditions $x_1(0) = x_2(0) = 0$ and $\dot{x}_1(0) = v_0$, such that an initial velocity is applied to only the LO. We simulated the response of the system for 1000 logarithmically spaced initial velocities for the range $v_0 \in [10^{-3}, 10^2]$ m/s, which provides a large enough range to probe the entire dynamics of the system. The response was simulated using ode45 in MATLAB for a time step of $10^{-4}$ s for a total time of 60 s. Additionally, the relative and absolute tolerances of solver were both set to $10^{-12}$. This combination of parameters ensures that the system can be integrated accurately at high initial velocities.

Just as in Section 2, we consider both the percent energy dissipated by each damper and the settling times of the LO. However, unlike the theoretical system, we now provide comparisons between the predictions of the model and the corresponding quantities from the experimental measurements. For the model, the dissipated energies are computed using Eqs. (9) and (10) with $\dot{x}_1$ and $\dot{x}_2$ instead of $\dot{\xi}_1$ and $\dot{\xi}_2$ and the dimensional values for each parameter. The resulting values are normalized by the initial mechanical energy, $\frac{1}{2}m_1v_0^2$, then multiplied by 100 to obtain the percent energy dissipated by each damper. For the experiments, the dissipated energies are computed using Eqs. (9) and (10) using the measured velocities and the dimensional parameters. However, the maximum mechanical energy is unknown in the experiments, since the excitation is due to an impulse supplied by a modal impact hammer. Thus, we normalize the dissipated energies by their sum and multiply the result by 100 to obtain the percent energy dissipated for each damper. The procedure applied to the experiments can also be applied to the simulation results, but the outcome is the same as normalizing by the initial mechanical energy. We depict the percent energy dissipated by each damper as functions of the maximum

![Fig. 13. Comparison between experimental and simulation response of the nonlinear system for an impact of 433.5 N for (a) LO and (b) NES.](image)

![Fig. 14. The percent energy dissipated by each damper for both the dimensional model and the experimental measurements.](image)
absolute velocity of the LO (i.e. the initial velocity in the case of the model) in Fig. 14. A good agreement is observed between the experimental and model results, which confirms the accuracy of the theoretical predictions. Although we were able to force the system with impacts higher than 1000 N, we found that the NES would jump out of its stable position between the two SH elements, resulting in a bifurcation that drastically changes the dynamics of the system. Thus, we excluded any forces where this occurred and were unable to achieve maximum LO velocities above 1 m/s.

As a second verification of the performance of the experimental NES, we computed the 50%, 10%, 5%, and 2% settling times for both the model and the experimental measurements. We depict the resulting comparison in Fig. 15. Although there is some minor disagreement between the predicted and measured settling times, there is good agreement in the trends of the two data sets. The agreement in trends confirms the accuracy of the theoretical results and the ability of an NES with an SH element to mitigate vibrations that occur at frequencies below its own linear (low energy) frequency.

5. Concluding remarks

This research focused on the mitigation of vibrations using a NES that had a linear, low-energy frequency above those vibrations using an SH strut element, which had a restoring force consisting of a linear segment, a plateau, and a stiffening segment. This combination of regimes caused the NES to undergo softening then hardening for increasing displacements, which gives it the ability to interact with vibrations occurring below its linear, low-energy frequency. The performance of the SH NES was compared with the performance of an NES with the same linear stiffness as the SH NES but with only a cubic nonlinear stiffness term. The comparison showed that the cubic NES was unable to interact with the LO whereas the SH NES was able to mitigate the motion of the LO within a certain regime of initial velocities. The theoretical results for the SH NES were validated experimentally with a comparable dimensional system. The results of this research demonstrate that NES elements can be incorporated into NESs to allow them to interact with vibrations that occur below their own linear, low-energy frequencies. Such NESs could find use in highly flexible structures with very low natural frequencies where conventional NESs or tuned-mass-dampers are unable to mitigate the vibrations. Furthermore, because the SH NES undergoes two regimes of nonlinear behavior (softening and hardening), it may improve upon the range of effectiveness of standard NESs allowing them to mitigate motion across wider ranges of excitation.


